MATH 320 NOTES

Review of some facts:
Let $T : V \to W$ be a linear transformation, V, W vector spaces over F.

(1) The dimension theorem: $\dim(V) = nullity(T) + rank(T)$.
(2) If V, W have the same finite dimension, T is one-to-one iff onto.
(3) T is invertible iff T is one–to-one and onto iff V, W are isomorphic.
(4) If $\dim(V) = n$, then $V \cong F^n$.
(5) If $\dim(V) = \dim(W)$ is finite, then $V \cong W$.

Notation and more facts:

(1) If $\dim(V) = n$ and β is a basis for V, then $[x]_\beta \in F^n$ is the coordinate vector.
(2) If $T : V \to W$ linear, V, W finite dimensional, β is a basis for V, γ is a basis for W, then $[T]_\beta^\gamma$ is the corresponding matrix representation and for any $x \in V$,
$$[T]_\beta^\gamma [x]_\beta = [T(x)]_\gamma.$$
(3) If A is a k by n matrix with entries in F, then multiplication by A, denoted by L_A is a linear transformation from $F^n \to F^k$.
(4) If A is n by n and e is the standard basis for F^n, then $[L_A]_e = A$.

Lemma 1. *An n by n matrix A is invertible iff L_A is invertible*

Lemma 2. *Let $\dim(V) = \dim(W) = n$, and $T : V \to W$ is a linear transformation, β any basis for V, γ any basis for W. Then T is invertible iff $[T]_\beta^\gamma$ is invertible.*

Section 2.5 Change of Coordinate Matrix.

Recall that $I_V : V \to V$ is the identity linear transformation, i.e. $I_V(x) = x$ for all $x \in V$.

Theorem 3. *Suppose V is finite dimensional vector space and β, α are two bases for V. The* **change of coordinate matrix from α to β**, *$Q = [I_V]_\alpha^\beta$ is such that:*

(1) *for every vector $x \in V$, $Q[x]_\alpha = [x]_\beta$,*
(2) *Q is invertible and $Q^{-1} = [I_V]_\beta^\alpha$,*
(3) *If $T : V \to V$ is a linear transformation, then $[T]_\alpha = Q^{-1}[T]_\beta Q$*

Definition 4. *Suppose that $A, B \in M_{n,n}(F)$. We say that A and B are* **similar** *if there is an invertible matrix Q such that $A = Q^{-1}BQ$. We write $A \sim B$*

As an exercise, prove the following:

Problem 1. *Being similar is an equivalence relation on matrices. I.e. it is reflexive, symmetric and transitive.*

Problem 2. *Suppose that $T : V \to V$ is a linear transformation, V is finite dimensional, and α, β are two bases for V. Then $[T]_\alpha$ and $[T]_\beta$ are similar matrices.*

Chapter 3. Section 3.1 Elementary matrices and matrix operations

Let A be an m by n matrix. The **elementary row operations** on A are:

(1) Interchanging two rows;
(2) Multiply a row by a nonzero scalar;
(3) Adding a multiple of one row to another.

Similar definition for **elementary column operations** – replace "row" by "column". We refer to the above by type (1), type (2), and type (3).

A is an **elementary matrix** if it obtained from I_n by doing **one** elementary row or column operation

Examples of elementary matrices:

(1) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

The next theorem captures the connection between elementary matrices and elementary operations. Informally, it says that doing an elementary row operation is equivalent to multiplying by an elementary matrix on the left. Similarly for column operation, but there the multiplication is on the right

Theorem 5. *Suppose that $A, B \in M_{m\times n}(F)$. Then B is obtained from A by doing one elementary row operation iff $B = EA$ for some elementar row matrix $E \in M_{m\times m}(F)$.*

Similarly, B is obtained from A by doing one elementary column operation iff $B = AE$ for some elementary n by n column matrix E.

Next we give some examples:

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Type (1). Suppose that $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $EA = \begin{pmatrix} -1 & 4 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Type (2). Suppose that $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $EA = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 20 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Type (3). Suppose that $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$

Then $EA = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 4 & 0 \\ -2 & 13 & 1 \end{pmatrix}$

Theorem 6. *Elementary matrices are invertible, and the inverse of an elementary matrix is elementary of the same type.*

Proof. We go over the three types. Suppose that E is an elementary matrix. We only consider row operations; the column case is similar.

(1) If E is of type (1) - interchanging rows i and j; then $E^{-1} = E$.
(2) If E is of type (2) - multiplying row i by the scalar c, then E^{-1} is multiplying the same row by $1/c$.
(3) E is of type (3) - adding a multiple of one row to another, say adding cR_i to R_j. Then E^{-1} is adding $-cR_i$ to R_j

In each of the above three cases, doing there two operations cancel out and takes you back to the identity matrix. So $EE^{-1} = I_n$

□

As a corollary we get that the product of elementary matrices is also an invertible matrix:

Corollary 7. *Suppose* $A = E_1 ... E_k$*, where each* E_i *is an elementary matrix. Then* A *is invertible.*

Proof. Let $B = E_k^{-1} ... E_1^{-1}$. Then $AB = BA = I_n$, and so $B = A^{-1}$.

□

Later we will show that the converse is also true i.e. every invertible matrix can be written as the product of elementary matrices.

Section 3.2 The Rank of a Matrix and Matrix Inverses

Definition 8. *Let $A \in M_{k\times n}(F)$. Define the* **rank of** *A, $rank(A) = rank(L_A)$. I.e. it is the dimension of the range of $L_A : F^n \to F^k$.*

Lemma 9. *Let $A \in M_{n\times n}(F)$. A is invertible iff $rank(A) = n$.*

Proof. A is invertible iff L_A is an isomorphism iff L_A is onto iff $rank(A) = \dim \operatorname{ran}(L_A) = \dim(F^n) = n$. □

Lemma 10. *Let $A \in M_{k\times n}(F)$. The rank of A equals the maximum number of its linearly independent columns i.e. the dimension of the column space of A.*

Proof. We have $L_A : F^n \to F^k$, and let $\beta = \{e_1, ..., e_k\}$ be the standard basis for F^k. Recall that for every i,

$$L_A(e_i) = Ae_i = a_i,$$

where a_i is the i-th column of A. Then,

$rank(A) = rank(L_A) = \dim(\operatorname{ran}(L_A)) = \dim Span(\{L_A(e_1), ..., L_A(e_k)\}) = \dim Span(\{a_1, ..., a_k\})$.

That is exactly the the maximum number of its linearly independent columns.

□

Exercise: Suppose that $T : V \to W$ is a linear transformation, $U : V \to V$ is invertible, and $L : W \to W$ is also invertible. Show that:

$$rank(TU) = rank(T) = rank(LT)$$

Theorem 11. *Let $A \in M_{k\times n}(F)$. Suppose that $P \in M_{k\times k}(F)$ is invertible, and $Q \in M_{n\times n}(F)$ is invertible. Then,*

$$rank(A) = rank(PA) = rank(AQ).$$

Proof. By the above exercise,

- $rank(PA) = rank(L_{PA}) = rank(L_P L_A) = rank(L_A) = rank(A)$
- $rank(AQ) = rank(L_{AQ}) = rank(L_A L_Q) = rank(L_A) = rank(A)$.

□

Corollary 12. *Elementary row operations preserve the rank.*

Next: computing the rank of a matrix with elementary row and column operations.

Theorem 13. *Let $A \in M_{k\times n}(F)$. By doing a finite number of row and column elementary operation, A can be transformed into a matrix of the form*

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}$$

where $r \le k, r \le n$, and $0_1, 0_2, 0_3$ are zero matrices. I.e. $D_{ii} = 1$ for all $i \le r$ and $D_{ij} = 0$ for all other entries.

Theorem 14. *Suppose A and D are as above. Then $A = BDC$, where B and C are the product of elementary matrices.*

Moreover if $k = n$, then

$$r = n \text{ iff } A \text{ is invertible.}$$

Proof. B corresponds to all the elementary row operations and C corresponds to all the elementary column operations.

Also suppose that $k = n$. Then A is invertible iff $rank(A) = n$ iff $r = n$ iff $D = I_n$ and so $A = BC$. □

Corollary 15. *A is invertible iff it is a product of elementary matrices.*

Next we list a couple of more facts relating to the **transpose** of a matrix. Recall that for $A \in M_{k\times n}(F)$, the transpose, $A^t \in M_{n\times k}(F)$ is the matrix where $(A^t)_{ij} = A_{ji}$. Recall also that $(AB)^t = B^t A^t$. The following lemma summarizes the properties that are preserved by taking the transpose.

Lemma 16.
(1) *If E is an elementary matrix, so is E^t.*
(2) *If A is invertible, then so is A^t.*
(3) *$rank(A) = rank(A^t)$,*
(4) *The row and column space of A have the same dimension.*

Proof. For item (1), it is an exercise to check that if E is an elementary row matrix, then E^t is an elementary column matrix of the same type. Similarly, if E is an elementary column matrix, then E^t is an elementary row matrix of the same type.

For item (2), let A be an invertible matrix. Then $A = E_1...E_k$ where each E_i is an elementary matrix. Then $A^t = (E_1...E_k)^t = E_k^t...E_1^t$, which is also a product of elementary matrices. So, A^t is also invertible.

For item (3), let $A \in M_{k\times n}(F)$. Write $A = BDC$ as in theorem 14. Then $A^t = (BDC)^t = C^t D^t B^t = C^t D B^t$. Both B and C are products of elementary matrices, so B^t and C^t are also products of elementary matrices. Then by theorem 13, $rank(A) = rank(A^t) = r$.

Item (4) follows by (3): $\dim(\text{row space}(A)) = \dim(\text{column space}(A^t)) = rank(A^t) = rank(A) = \dim(\text{column space}(A))$.

□

Computing inverses:

Suppose A is an invertible n by n matrix. Take the n by $2n$ matrix $(A|I_n)$. Do elementary row operations until you get a matrix of the form $(I_n|B)$. Then $B = A^{-1}$.

Section 3.3 Systems of linear equations

Suppose that $A \in M_{k\times n}(F), b \in F^k, x \in F^n$. Consider the following system of k linear equations and n unknowns:

$$Ax = b,$$

where A, b are given and we want to solve for x.

In this section we will go over criteria to determine when solutions exists and if the solution is unique.

Definition 17. *The system $Ax = b$ is called* **homogeneous** *is $b = \vec{0}$.*

First we investigate when homogeneous systems have a nontrivial solution. (Note that $\vec{0}$ is always a solution for the system $Ax = \vec{0}$).

Theorem 18. *Let K denote all solutions to the system $Ax = \vec{0}$, where $A \in M_{k\times n}(F)$. Then $K = \ker(L_A)$, and so it is a subspace of F^n; $\dim(K) = n - rank(A)$.*

Proof. $L_A : F^n \to F^k$, and for every $s \in F^n$, s is a solution to $Ax = \vec{0}$ iff $As = L_A(s) = \vec{0}$ iff $s \in \ker(L_A)$. By the dimension theorem

$$\dim(K) = n - rank(A).$$

□

As a corollary we can show that a homogenous system with more unknowns that equations, always has a nontrivial (i.e. nonzero) solution.

Corollary 19. *If $A \in M_{k\times n}(F)$, and $k < n$, then the system $Ax = \vec{0}$ always has a nontrivial solution.*

Proof. Let K be all solutions. Since $rank(A) \leq k < n$, it follows that $\dim(K) = n - rank(A) > 0$, so K has nonzero vectors. □

Next we investigate solutions to nonhomogeneous systems. We will see that although the set os solutions is not a subspace like in homogeneous systems, it is the next best thing.

Theorem 20. *Suppose $A \in M_{k\times n}(F)$, and K is the set of solutions to the system $Ax = b$. Denote K_H to be the set of solutions to the corresponding homogeneous system $Ax = \vec{0}$. Let s be any solution to $Ax = b$. Then,*

$$K = \{s\} + K_H = \{s + k \mid k \in K_H\}$$

Proof. Let s be any solution to $Ax = b$. First we show that $\{s\} + K_H \subset K$. Suppose that $k \in K_H$. We want to show that $s + k \in K$, i.e. that $s + k$ is a solution to $Ax = b$. To do that, simply check

$$A(s + k) = As + Ak = As + \vec{0} = b.$$

So $s + k \in K$.

For the other direction, suppose that $w \in K$; we have to show that $w \in \{s\} + K_H$. Since both w and s are solution to $Ax = b$, we have $Aw = b = As$, and so $Aw - As = \vec{0}$.

Simplifying the left hand side, we get

$$A(w - s) = \vec{0} \Rightarrow w - s \in K_H.$$

Setting $k := w - s$, we get that $w = s + k \in \{s\} + K_H$

□

Finally, we investigate for which vectors b, the system $Ax = b$ is **consistent**, which means that there is a solution.

Lemma 21. *Suppose that $A \in M_{n\times n}(F)$ is invertible. Then for any $b \in F^n$, the system $Ax = b$ has exactly one solution, given by $A^{-1}b$.*

Proof. Multiply by A^{-1} on both sides of the equation $Ax = b$. □

Lemma 22. *Suppose that $Ax = b$ is a system of linear equations, $A \in M_{k\times n}(F)$. Then the system is consistent iff $rank(A) = rank(A|b)$. Here $(A|b)$ the the augmented matrix obtained by adding b as an additional column to A.*

Proof. Let $a_1, ..., a_n$ be the columns of A. Recall that $\mathrm{ran}(L_A) = Span(\{a_1, ..., a_n\})$, and also that the rank of a matrix equals the dimension of the column space.

The system is consistent iff there is some $s \in F^n$, such that $As = b$ iff
$b \in \mathrm{ran}(L_A) = Span(\{a_1, ..., a_n\})$ iff
$\dim Span(\{a_1, ..., a_n\}) = \dim Span(\{a_1, ..., a_n, b\})$ iff
$rank(A) = rank(A|b)$.

□